Connection between tiles, weak tiles and spectral sets

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Basic definitions

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Early investigation concentrated on the case of the Euclidean spaces, which have a strong relation with the case of finite abelian groups.

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- Conjecture holds for the union of two intervals (Łaba)
- Conjecture holds for convex bodies in $\mathbb{R}^2$ (Iosevich, Katz, Tao).

Finite abelian groups

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The exponential functions are characters or in other words irreducible representations. These functions are parametrized by the elements of G and they do form a group where the multiplication is the pointwise multiplication. Moreover, this group $\hat{G}$ is isomorphic to G so the spectrum can also considered as a subset of G. We will index the representations by the elements of G.

Spectral pair

(T, Λ) is a *spectral pair*, where $T \subset G, \Lambda \subset \hat{G}$, if

$$\chi_{\lambda_1 - \lambda_2}(T) = \sum_{t \in T} \chi_{\lambda_1 - \lambda_2}(t) = \hat{1}_T(\lambda_1 - \lambda_2) = 0.$$

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The 3 dimensional case of both directions of the conjecture is disproved [4, 2]. (Kolountzakis-Matolcsi and Farkas, Matolcsi, Móra)

Some results for cyclic groups

Spectral-tile direction:

- $\mathbb{Z}_{p^m q^n}$, where $m \leq 6$ and $n \leq 9$ or $p^{m-2} < q^4$, Malikiosis, [9],
- $\mathbb{Z}_{pqr}$, Shi, [10],
- $\mathbb{Z}_{pqrs}$, Kiss, Malikiosis, S., Vizer [3],
- $\mathbb{Z}_{p^2 qr}$, S. [12].

Tile-spectral direction:

- $\mathbb{Z}_{p^k q^l}$, Łaba, [5],
- $\mathbb{Z}_{np}$, where n is square-free, Malikiosis, [9],
- $\mathbb{Z}_{(pqr)^2}$, Łaba, Londner. [6, 7]
- $p_1^{n_1} p_2^{n_2} p_3^{n_3}$ with $p_1 > p_2^{n_2} p_3^{n_3}$ and $p_1^{n_1} p_2^2 p_3^2 p_4^2$ with $p_1 > p_2 p_3 p_4$ Łaba, Londner [8].

Convex sets, Fuglede's conjecture

- Iosevich Katz Tao: Convex bodies in $\mathbb{R}^2$.
- Greenfeld, Lev: Convex polytopes in $\mathbb{R}^3$.
- Lev, Matolcsi: Convex bodies
  - 'It has long been known (Venkov, McMullen) that a convex body that tiles by translations must be a polytope, and that it admits a face-to-face tiling by a lattice translation set Λ and therefore has a spectrum given by the dual lattice Λ^*'.
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  - Matolcsi and Lev introduced the notion weak tile.
- A tiles G with B if and only if $1_A * 1_B = 1_G$.
- A tiles G weakly if there is a function $w\colon G \to \mathbb{R}^{\geq 0}$ with $w(0) = 1$ and $1_A * w = 1_G$.

Question on weak tiles

- If A tiles G, then A weakly tiles G.
- If A is spectral in G, then A weakly tiles G. This is the novelty of Lev and Matolcsi. In the case of a spectral pair (A, B) in a finite group G, a weak tile complement is equal to $w = \frac{1}{|B|^2} \hat{1}_B * \hat{1}_{-B}$.

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- Since Tao proved the existence of a spectral set which is not a tile we know that there is a weak tile which is not a tile so this is indeed a new definition.
- Is it really a new one?
- Kolountzakis, Lev and Matolcsi asked whether there is a weak tile which is neither spectral nor a tile.

Lonely weak tile

Theorem (Kiss, Londner, Matolcsi, S.)

There is a set in a suitable chosen finite abelian group which is a weak tile but which is neither a tile nor spectral.

- Tao's construction: There is a Hadamard matrix of size 12. Take a basis in $\mathbb{Z}_2^{12}$. This is a spectral set which is not a tile since $12 \nmid 2^{12}$.
  Similar construction can be carried out in $\mathbb{Z}_3^6$ and we may reduce the dimension by 1.
- For every odd p there is a similar example B in $H := \mathbb{Z}_p^4$.
- Kolountzakis-Matolcsi construction: Take a 'basis' in $\mathbb{Z}_6^5$ and add 0. This is a set that tiles $\mathbb{Z}_6^5$ and does not have a universal spectrum.
  This allows us to contruct a set A in $\mathbb{Z}_6^5 \times \mathbb{Z}_q$ which is a tile but not spectral (q is large enough, not necessarily a prime).

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  - This is not a tile (as we will se later).
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    We could not prove it and it is seems to be a spectral set.
- **A better idea is the following:**

$$P_t := \cup_{a \in A} B + t(a).$$

  - This is still not a tile.
  - Is it spectral? It does depend on the choice of t.

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$$\hat{1}_{P_t}(\gamma, \rho) = \hat{1}_B(\rho) \sum_{a \in A} \gamma(a)\rho(t_a). \tag{1}$$

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Let us specify t, which is a map from A to $H = \mathbb{Z}_p^4$:

$$t(0_{\mathbb{Z}_6^5}, k) = v_k \ \text{ for } \ 1 \le k \le 4, \ \text{ and } \ t(a,j) = 0_H \ \text{ otherwise.} \qquad (2)$$

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$$\hat{1}_{P_t}(\gamma, \rho) = \hat{1}_B(\rho) \left(\hat{1}_A(\gamma) - \sum_{i=1}^{4} \gamma(a_i)(1 - \rho(v_i)) \right). \tag{3}$$

Proposition

Let P_t be as above and assume q is a prime. Assume that $\hat{1}_B(\rho) \neq 0$, and $\rho \neq 0$, and the $\mathbb{Z}_q$-component of $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_6^5 \times \mathbb{Z}_q$ satisfies $\gamma_2 \neq 0$. Then $\hat{1}_{P_t}(\gamma, \rho) \neq 0$.

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A **fake message** of this construction. The following strategy might not work: Every spectral set tiles weakly. In order to prove the spectral-tile direction (for certain finite groups) of Fuglede's conjecture try to prove that every weak tile is a tile.

Cyclotomic divisibility

- We would like to know when the sum of M'th roots of unities is equal to zero. This has been described.
- Let $p_1, p_2, \dots p_k$ be the different prime divisors of M. Clearly, the sum of all p_i'th roots of unities is 0.
- We identify the set of roots of unities with $\mathbb{Z}_M$. Then 'subgroup sums' are zero and then 'coset sums' are also 0. Linear combination of characteristic functions of cosets will also vanish at the corresponding representation.

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- We identify the set of roots of unities with $\mathbb{Z}_M$. Then 'subgroup sums' are zero and then 'coset sums' are also 0. Linear combination of characteristic functions of cosets will also vanish at the corresponding representation.
- Classical result (de-Bruijn-Schoenberg, Lam-Leung) says that there is no other way of getting 0 and if M has at most two different prime divisors ($k = 2$), then multisets are non-negative linear combinations of these building blocks.

A generalisation

We use polynomial notation so instead of saying that certain representation is a root of the Fourier transform we write $\Phi_k \mid A(x) = \sum_{a\in A} x^a$, so this cyclotomic polynomial divides the mask polynomial of the set A.

Proposition (Kiss, Łaba, Marshall, S.; Long fiber decomposition)

Let $M = \prod_{i=1}^k p_i^{n_i}$, and let $N|M$ satisfy $N = \prod_{i=1}^k p_i^{n_i-\alpha_i+1}$ with $1 \le \alpha_i \le n_i$. Let $A \in \mathcal{M}(\mathbb{Z}_M)$, and assume that $\Phi_L(X) \mid A(X)$ for each $N \mid L \mid M$. Then, there exist polynomials $P_i(X) \in \mathbb{Z}[X]$ such that

$$A(X) = P_1(X)F_{1,\alpha_1}(X) + \cdots + P_k(X)F_{k,\alpha_K}(X) \mod X^M - 1.$$

Moreover, if $A \in \mathcal{M}^+(\mathbb{Z}_M)$ and $K = 2$, then we may assume that the polynomials $P_1(X)$ and $P_2(X)$ each have non-negative coefficients.

Playing (T2) against (T1)

- The so-called Coven-Meyerowitz conjecture says that tiles of finite cyclic groups satisfy two properties (T1) and (T2).
- Sets satisfying (T1) and (T2) tile and spectral (Łaba).
- Every tile satisfy (T1).
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- Strategy is to assume $A \bigoplus B = \mathbb{Z}_M$ and assume A does not satisfy (T2). Try to prove that B is too large and does not satisfy (T1).
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- We further assumed that B has all (T2) divisors and by the previous assumptions it has an extra divisor. Does it cause an extra size increase?
- We tried to apply this idea but the intermediate statements fail because our long-fiber shifting constructions.

A conjecture of Greenfeld and Lev

- It is easy to see that $A \times B$ is a tile if and only A and B are tiles.
- Does the same hold for spectral sets as well? This question was raised a few times by Nir Lev.
  - Greenfeld, Lev: if A is an interval B is a subset of $\mathbb{R}^{n-1}$ and $A \times B$ is spectral, then A and B are spectral.
  - Kolountzakis extended this result to the case, when A is the union of two intervals.
  - Greenfeld, Lev proved the analogous result when A is a convex polygon and conjectured that this holds in general.
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Theorem (S.)

Let G be a finite abelian group of order n and let $D = \{(g,g) \mid g \in G\}$ denote the diagonal subgroup of $G \times G$.

1. *Let $P = \{(a_i, b_i) \mid i = 1, \ldots, n\}$ be a subset of $G \times G$. Then (P, D) is a spectral pair if and only if $\{a_i + b_i \mid i = 1, \ldots, n\} = G$.*
2. *Let A and B subsets of G with $|A| * |B| = |G|$. Then A tiles with B if and only if $(A \times B, D)$ is spectral in $G \times G$.*

Questions, suggestions

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- Combine the ideas of the previous two papers.
- Can we use this new idea to construct $S - T(\mathbb{R}^2)$ counterexample?
- Is there another meaningful subset E of $G \times G$ such that if (S, E) is a spectral pair, then S has some nice property.

Gabor basis

Theorem (Iosevich, Kolountzakis, Lyubarskii, Mayeli, Pakianathan)

Suppose that $E \subset \mathbb{Z}_p^d$*, where* p *is a prime. Then* $\left\{ \frac{1}{|E|^{-1/2}} 1_E(x-a)\chi_b(x) \right\}_{a \in A, b \in B}$ *is an orthonormal basis for* $L^2(\mathbb{Z}_p^d)$ *if and only if* (E, B) *is a spectral pair and* (E, A) *is a tiling pair.*

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The same can be formulated (and holds) for any finite abelian groups.

Assume $\{f(x-g)\chi_g(x)\}_{g \in G}$ is an orthogonal basis in $L^2(G)$.

- What can we say about f?
- Is there such an f for every finite abelian group G?
- What if we impose the condition that f is a characteristic function?

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